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Probabilistic and Average Linear Widths in L_∞-Norm with Respect to r-fold Wiener Measure

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We show that for r-fold Wiener measure, the probabilistic and average linear widths in the L_∞-norm are proportional to $n^{-(r+1/2)}\sqrt{\ln n/\delta}$ and $n^{-(r+1/2)}\sqrt{\ln n}$, respectively. © 1996 Academic Press, Inc.

1. INTRODUCTION

We study *probabilistic* linear (n, δ)-widths and *average* linear n-widths for L_∞-approximation of functions that are distributed according to the r-fold Wiener measure. As the clasical n-widths (see, e.g., [9]); probabilistic and average widths quantify the error of best approximating operators. However, in the classical approach, the errors are defined by their worst case with respect to a given class (typically a unit ball of the underlying space). In the probabilistic approach, the errors are defined by the worst case performance on a subset of measure at least $1-\delta$, and in the average case approach, they are defined by their expectations, both with respect to a given probability measure.

The study of probabilistic and average widths has been suggested only recently (see, e.g., [8, 13]) and relatively few results have been obtained so far (see, e.g., [1, 2, 4–7, 10, 12, 13]). These include results on probabilistic and average Kolmogorov widths in L_q-norm for any $q \leqslant \infty$ and on probabilistic and average linear widths in L_q-norm for finite q. In both cases, the underlying space of function is the $C^r[0, 1]$ space equipped with

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the r-fold Wiener measure. More specifically, the upper bounds on the average widths follow from [11] for $q<\infty$ and [10] for $q=\infty$. The asymptotic lower bounds on average Kolmogorov widths with arbitrary q and on average linear widths with finite q are mainly due to [4–6]. The results concerning probabilistic Kolmogorov and linear widths are also due to [4–6]. Our result concerning the probabilistic and average linear widths for $q=\infty$ provides the last missing piece as far as the probabilistic nd average linear widths with r-fold Wiener measures are concerned. Thus, denoting probabilistic Kolmogorov and linear (n,δ)-widths by $d^{(p)}_{n,\delta}(C^r, L_q, \omega_r)$ and $\lambda^{(p)}_{n,\delta}(C^r, L_q, \omega_r)$, and average Kolmogorov and linear n-widths by $d^{(a)}_n(C^r, L_q, \omega_r)$ and $\lambda^{(a)}_n(C^r, L_q, \omega_r)$, respectively, we conclude that

$$d^{(p)}_{n,\delta}(C^r,L_q,\omega_r) \asymp n^{-(r+1/2)}\sqrt{1+n^{-1}\ln(1/\delta)}, \qquad 1\leqslant q\leqslant\infty,$$

$$\lambda^{(p)}_{n,\delta}(C^r,L_q,\omega_r) \asymp \begin{cases} n^{-(r+1/2)}\sqrt{1+n^{-\min\{1,2/q\}}\ln(1/\delta)}, & 1\leqslant q<\infty,\\ n^{-(r+1/2)}\sqrt{\ln(n/\delta)}, & q=\infty,\end{cases}$$

$$d^{(a)}_n(C^r,L_q,\omega_r) \asymp n^{-(r+1/2)}, \qquad 1\leqslant q\leqslant\infty,$$

$$\lambda^{(a)}_n(C^r,L_q,\omega_r) \asymp \begin{cases} n^{-(r+1/2)}, & 1\leqslant q<\infty,\\ n^{-(r+1/2)}\sqrt{\ln n}, & q=\infty.\end{cases}$$

It is interesting to note that for finite q, the average Kolmogorov and average linear n-widths are equal modulo multiplicative constants. We have also equality between probabilistic Kolmogorov and linear (n,δ)-widths for $q\leqslant 2$. For such values of q, linear approximation opertors are (modulo a constant) as good as nonlinear operators. The difference is only for $q=\infty$ (for average widths) and for $q>2$ (for probabilistic widths); however, then linear operators lose to optimal nonlinear operators only by a factor of $\sqrt{\ln n}$ and $n^{(1/2-1/q)_+}$, respectively.

The paper is organized as follows. Basic definitions and the main result are provided in Section 2. The proof of the result is in Section 3.

2. Main Result

For a nonnegative integer r, let C^r be the space of r times continuously differentiable functions defined on $[0,1]$. Recall that the corresponding Kolmogorov and linear n-widths are defined respectively by

$$d_n(C^r, L_q) = \inf_{T\in\Lambda_n}\ \sup_{f\in B(C^r)} \|f-T(f)\|_q, \tag{1}$$

$$\lambda_n(C^r, L_q) = \inf_{T\in\mathscr{L}_n}\ \sup_{f\in B(C^r)} \|f-T(f)\|_q, \tag{2}$$

where $B(C^r)$ is the unit ball in C^r, Λ_n is the class of all (not necessarily linear) operators $T: B(C^r) \to L_q$ whose range is contained in an n-dimensional subspace of L_q, and $\mathscr{L}_n$ is the class of all linear operators from Λ_n.

Let μ be a probability measure defined on the Borel σ-field of C^r. Given $\delta \in [0, 1]$, the corresponding *probabilistic Kolmogorov* and *probabilistic linear* (n, δ)-*widths* are defined by

$$d^{(p)}_{n,\delta}(C^r, L_q, \mu) = \inf_G \inf_{T \in \Lambda_n} \sup_{f \in G} \|f - T(f)\|_q, \tag{3}$$

$$\lambda^{(p)}_{n,\delta}(C^r, L_q, \mu) = \inf_G \inf_{T \in \mathscr{L}_n} \sup_{f \in G} \|f - T(f)\|_q. \tag{4}$$

The first infima are taken with respect to all measurable sets $G \subset C^r$ with $\mu(G) \geqslant 1 - \delta$.

The *average Kolmogorov* and *average linear n-widths* are defined by

$$d^{(a)}_n(C^r, L_q, \mu) = \inf_{T \in \Lambda_n} \mathrm{E}_\mu(\|f - T(f)\|_q), \tag{5}$$

$$\lambda^{(a)}_n(C^r, L_q, \mu) = \inf_{T \in \mathscr{L}_n} \mathrm{E}_\mu(\|f - T(f)\|_q). \tag{6}$$

Here E_μ denotes the expectation with respect to μ, i.e.,

$$\mathrm{E}_\mu(\|f - T(f)\|_q) = \int_{C^r} \|f - T(f)\|_q \, \mu(df).$$

Obviously,

$$\begin{aligned} d^{(a)}_n(C^r, L_q, \mu) &= \int_0^1 d^{(p)}_{n,\delta}(C^r, L_q, \mu)\, d\delta, \\ \lambda^{(a)}_n(C^r, L_q, \mu) &= \int_0^1 \lambda^{(p)}_{n,\delta}(C^r, L_q, \mu)\, d\delta. \end{aligned} \tag{7}$$

In what follows we assume that μ equals the r-fold Wiener measure ω_r. For basic properties of ω_r, see, e.g., [3]. Here we only mention that ω_r is a zero mean Gaussian measure with the covariance function

$$\mathrm{E}_{\omega_r}(f(x)\, f(y)) = \int_0^1 \frac{(x-t)^r_+\, (y-t)^r_+}{(r!)^2}\, dt$$

and that $\omega_r(A) = \omega_0(D^r A)$, where ω_0 is the classical Wiener measure on the space C^0 and D^r is the differential operator, $D^r f = f^{(r)}$.

As mentioned in Introduction, the probabilistic and average Kolmogorov widths have been found for any q, and the probabilistic and average linear widths have been found only for finite q. The following theorem deals with probabilistic and average linear widths for $q=\infty$.

THEOREM 1. *For every r and $\delta\in(0,\frac{1}{2})$,*

$$\lambda^{(p)}_{n,\delta}(C^r, L_\infty, \omega_r) \asymp n^{-(r+1/2)}\sqrt{\ln(n/\delta)},$$
$$\lambda^{(a)}_{n}(C^r, L_\infty, \omega_r) \asymp n^{-(r+1/2)}\sqrt{\ln n}. \tag{8}$$

Actually, the proof of Theorem 1 provides another result concerning linear widths for finite dimensional spaces. Let l^m_∞ denote the space $\mathbb{R}^m$ equipped with the maximum norm, and let γ denote zero mean normal distribution with the identity covariance matrix, $\gamma=\mathcal{N}(0,I)$.

THEOREM 2. *Let $m>2n$ and $\delta\in(0,\frac{1}{2})$. Then*

$$\lambda^{(p)}_{n,\delta}(\mathbb{R}^m, l^m_\infty, \gamma) \asymp \sqrt{\ln((m-n)/\delta)}, \qquad \lambda^{(a)}_{n}(\mathbb{R}^m, l^m_\infty, \gamma) \asymp \sqrt{\ln(m-n)}. \tag{9}$$

3. PROOF

Due to (7), we only need to show the equality concerning the probabilistic widths. We begin with few auxiliary lemmas.

LEMMA 1. *Let $m>2n$. Let T be any operator in $\mathbb{R}^m$ whose range is contained in an n-dimensional subspace. For $i=1,\dots,m$, let $g_i=e_i-T^*(e_i)$, where e_i is the ith unit vector. Then there are distinct indices $i_1,\dots,i_{m-2n}$ such that*

$$\operatorname{dist}\{g_{i_{s+1}}, \operatorname{lin}\{g_{i_1},\dots,g_{i_s}\}\} \geqslant 1/\sqrt{2}$$

for all $s=0,\dots,m-2n-1$.

Proof. It is known (see, e.g., [9]) that the following Kolmogorov n-widths equal

$$d_n(\operatorname{conv}(e_1,\dots,e_m), l^m_2)=\sqrt{(m-n)/m}. \tag{10}$$

Therefore, there is i_1 such that

$$\operatorname{dist}\{g_{i_1},\{0\}\} = \|e_{i_1}-T^*(e_{i_1})\|_2 \geqslant \sqrt{(m-n)/m}.$$

Assume by induction that $i_1,\dots,i_s$ ($s\leqslant m-2n-1$) exist. Consider the index sets $I=\{i_1,\dots,i_s\}$ and $I'=\{1,\dots,m\}\setminus I$, and the following operator $P\colon l^m_2\to l^{m-s}_2$, $Px=(x_i)_{i\in I'}$. From (10) it follows that

$$d_n(\operatorname{conv}\{Pe_i : i\in I'\}, l^{m-n}_2) \geqslant \sqrt{(m-s-n)/(m-s)}.$$

Therefore, there is $i_{s+1} \in I'$ for which

$$\operatorname{dist}\{Pe_{i_{s+1}}, PL\} \geqslant \sqrt{(m-s-n)/(m-s)},$$

where $L = \operatorname{Im} T^*$. Since $Pg_i = -PT^*e_i \in PL$ for any i, we have

$$\begin{aligned}\operatorname{dist}\{g_{i_{s+1}}, \operatorname{lin}\{g_{i_1}, ..., g_{i_s}\}\} &\geqslant \operatorname{dist}\{Pg_{i_{s+1}}, \operatorname{lin}\{Pg_{i_1}, ..., Pg_{i_s}\}\} \\ &\geqslant \operatorname{dist}\{Pe_{i_{s+1}}, PL\} \geqslant \sqrt{(m-s-n)/(m-s)}.\end{aligned}$$

Since $s \leqslant m-2n$, we have $(m-s-n)/(m-n) \geqslant \frac{1}{2}$. This completes the proof. ▌

Let $k = m-2n$, and let $r_1, ..., r_k \in \mathbb{R}^m$. By $G_a(r_1, ..., r_k)$ we denote the following polytop in $\mathbb{R}^m$:

$$G_a(r_1, ..., r_k) = \{x \in \mathbb{R}^m : \max_{1 \leqslant i \leqslant k} |\langle r_i, x\rangle| \leqslant a\}.$$

LEMMA 2. *Let $h_s = g_{i_s}$ for $s = 1, ..., k$. Then*

$$\gamma(x : \|x - Tx\|_\infty \geqslant a) \geqslant 1 - \gamma(G_a(h_1, ..., h_k)).$$

Proof. Since

$$\begin{aligned}\|x - Tx\|_\infty &= \max_{1 \leqslant i \leqslant m} |\langle e_i, x - Tx\rangle| \\ &= \max_{1 \leqslant i \leqslant m} |\langle e_i - T^*e_i, x\rangle| \\ &\geqslant \max_{1 \leqslant i \leqslant k} |\langle h_i, x\rangle|,\end{aligned}$$

we have $\gamma(x : \|x - Tx\|_\infty \geqslant a) \geqslant \gamma(x : \max_{1 \leqslant i \leqslant k} |\langle h_i, x\rangle| \geqslant a) = 1 - \gamma(G_a(h_1, ..., h_k))$, which completes the proof. ▌

LEMMA 3. *For*

$$a = \sqrt{\frac{1}{2} \ln \frac{m-2n}{2\delta}}$$

we have

$$\gamma(G_a(h_1, ..., h_k)) \leqslant \left(1 - \frac{2\delta}{m-2n}\right) \gamma(G_a(h_1, ..., h_{k-1})).$$

Proof. Due to rotational invariance of γ we can assume without loss of generality that

$$h_1, ..., h_{k-1} \in \operatorname{lin}\{e_1, ..., e_{k-1}\}, \qquad h_k \in \operatorname{lin}\{e_1, ..., e_k\}.$$

For $x \in \mathbb{R}^m$, let $x' = (x_1, ..., x_{k-1})$. Then

$$\gamma(G_a(h_1, ..., h_k)) = (2\pi)^{-k/2} \int_{G_a(h_1, ..., h_{k-1})} \exp(-\|x'\|_2^2/2) \times \int_{|\langle x', h_k' \rangle + x_k h_{kk}| \leqslant a} \exp(-x_k^2/2)\, dx_k\, dx'$$

$$\leqslant (2\pi)^{-(k-1)/2} \int_{G_a(h_1, ..., h_{k-1})} \exp(-\|x'\|_2^2/2)\, dx' (2\pi)^{-1/2} \times \int_{|th_{kk}| \leqslant a} \exp(-t^2/2)\, dt$$

$$= \gamma(G_a(h_1, ..., k_{k-1}))(2\pi)^{-1/2} \int_{|th_{kk}| \leqslant a} \exp(-t^2/2)\, dt.$$

From Lemma 1 it follows that $|h_{kk}| = \text{dist}\{h_k, \text{lin}\{h_1, ..., h_{k-1}\}\} \geqslant 1/\sqrt{2}$. Therefore,

$$\gamma(G_a(h_1, ..., h_k)) \leqslant \gamma(G_a(h_1, ..., k_{k-1}))(2\pi)^{-1/2} \int_{|t| \leqslant a\sqrt{2}} \exp(-t^2/2)\, dt. \tag{11}$$

To complete the proof we need only to show that

$$(2\pi)^{-1/2} \int_{|t| \leqslant a\sqrt{2}} \exp(-t^2/2)\, dt \leqslant 1 - \frac{2\delta}{m - 2n} \tag{12}$$

for sufficiently large $(m-2n)/(2\delta)$.

For this end, we use the fact that

$$\sqrt{\frac{2}{\pi}} \int_x^\infty e^{-t^2/2}\, dt \geqslant \frac{c}{x} e^{-x^2/2} \qquad \forall c \in (0, \sqrt{2/\pi}),\ \forall x \geqslant \sqrt{\frac{c}{\sqrt{2/\pi} - c}}, \tag{13}$$

which follows from the fact that $f(x) = \sqrt{2/\pi} \int_x^\infty e^{-t^2/2}\, dt - ce^{-x^2/2}/x$ has a negative derivative for such values of c and x and the fact that $f(\infty) = 0$.

Using $c = 1/\sqrt{\pi}$ and $x = a\sqrt{2}$, we have $x \geqslant \sqrt{c/(\sqrt{2/\pi} - c)}$ whenever $(m-2n)/(2\delta) \geqslant \exp(1/(\sqrt{2} - 1))$. Moreover, $c/x \geqslant \sqrt{2\delta/(m-2n)}$ whenever $(m-2n)/(2\delta) \geqslant \pi \ln((m-n)/(2\delta))$. Thus (13) holds when

$$\frac{m-2n}{2\delta} \geqslant \max\left\{\pi \ln \frac{m-2n}{2\delta},\ e^{1/(\sqrt{2}-1)}\right\}.$$

This completes the proof of Lemma 3. ▮

We are ready to prove Theorem 2.

Proof of Theorem 2. Applying Lemma 3 $k=m-2n$ times, we get

$$\gamma(G_a(h_1, ..., h_k)) \leqslant \left(1-\frac{2\delta}{k}\right)^k.$$

Hence, Lemma 2 yields

$$\gamma(x : \|x-Tx\|_\infty \geqslant a) \geqslant 1-\left(1-\frac{2\delta}{k}\right)^k \geqslant \delta.$$

Since T is arbitrary, this proves that

$$\lambda_{n\delta}^{(p)}(\mathbb{R}^m, l_\infty^m, \gamma) \geqslant a \asymp \sqrt{\ln((m-n)/\delta)}.$$

To prove equality observe that $\gamma(Q) \geqslant 1-\delta$ for

$$Q=\{x \in \mathbb{R}^m : \max_{1 \leqslant i \leqslant m-n} |x_i| \leqslant \sqrt{\ln((m-n)/\delta)}\}.$$

This means that for the orthogonal projection operator T on $\operatorname{lin}\{e_{m-n+1}, ..., e_m\}$,

$$\sup_{x \in Q} \|x-Tx\|_\infty \leqslant \sqrt{\ln((m-n)/\delta)}.$$

This proves that $\lambda_{n,\delta}^{(p)} \asymp \sqrt{\ln((m-n)/\delta)}$, as claimed, and completes the proof of Theorem 2.

We are ready to prove Theorem 1.

Proof of Theorem 1. We begin with the lower bound:

$$\lambda_{n,\delta}^{(p)}(C^r, L_\infty, \omega_r) \geqslant c_r n^{-(r+1/2)} \sqrt{\ln(n/\delta)} \tag{14}$$

for a positive constant c_r. For this end, we consider the inverse function of probabilistic widths. That is, given n, let

$$e_n(\varepsilon; C^r, L_\infty, \omega_r) := \inf_{T \in \mathscr{L}_n} \omega_r(f \in C^r : \|f-T(f)\|_\infty \geqslant \varepsilon)$$

for $\varepsilon \geqslant 0$. Obviously, $e_n(\varepsilon; C^r, L_\infty, \omega_r)=\delta$ for $\varepsilon=\lambda_{n,\delta}^{(p)}(C^r, L_\infty, \omega_r)$.

Take now $m=2n$ and $a_i=i/m$ for $i=1, ..., m$. Let $\mu_{r,0}=\omega_r(\cdot \mid N(f)=0)$ be the conditional measure with $N(f)=[f^{(j)}(a_i)=0 : 0 \leqslant j \leqslant r, 1 \leqslant i \leqslant m]$. Since ω_r is Gaussian,

$$\begin{aligned} e_n(\varepsilon; C^r, L_\infty, \omega_r) &\geqslant e_n(\varepsilon; C^r, L_\infty, \mu_{r,0}) \\ &\geqslant \inf_{T \in \mathscr{L}_n} \mu_{r,0}(f \in C^r : \max_{1 \leqslant i \leqslant m} |f(t_i)-T(f)(t_i)| \geqslant \varepsilon), \end{aligned}$$

where $t_i = (a_{i-1} + a_i)/2 = (2i-1)/(2m)$. Let $M(f) = [f(t_1), ..., f(t_m)]$, let $\mu^*_{r,0} = \mu_{r,0} M^{-1}$ be the induced probability on $\mathbb{R}^m$, and let $v = \mu_{r,0}(\cdot \mid M(f) = 0)$ be the conditional probability with $M(f) = 0$. Letting $s_y(x) = \sum_{i=1}^m y_i s_j(x)$ be the mean element of $\mu_{r,0}(\cdot \mid M(f) = y)$ $(y = (y_1, ..., y_m) \in \mathbb{R}^m)$, we have that any f can be represented as $f = s_y + h$, where y is distributed according to $\mu^*_{r,0}$, h is distributed according to v, and y and h are independent. Since $s_y(t_i) = y_i$, we have that for an arbitrary $T \in \mathscr{L}_n$

$$\mu_{r,0}(f \in C^r : \max_{1 \leqslant i \leqslant m} |f(t_i) - T(f)(t_i)| \geqslant \varepsilon)$$

$$= (\mu^*_{r,0} \times v)((y, h) \in \mathbb{R}^m \times C^r : \max_{1 \leqslant i \leqslant m} |y_i - T(s_y)(t_i) + h(t_i) - T(h)(t_i)| \geqslant \varepsilon)$$

$$\geqslant \mu^*_{r,0}(y \in \mathbb{R}^m : \max_{1 \leqslant i \leqslant m} |y_i - A(y)| \geqslant \varepsilon)$$

with the matrix $A = (a_{i,j})$ given by $a_{i,j} = T(s_j)(t_i)$. (The inequality above follows from the fact that v is zero mean Gaussian.)

Since the rank of T does not exceed n, so does the rank of A. This implies that $e_n(\varepsilon; C^r, L_\infty, \omega_r) \geqslant e_n(\varepsilon; \mathbb{R}^m, l^m_\infty, \mu^*_{r,0})$, or equivalently, that

$$\lambda^{(p)}_{n,\delta}(C^r, L_\infty, \omega_r) \geqslant \lambda^{(p)}_{n,\delta}(\mathbb{R}^m, l^m_\infty, \mu^*_{r,0}). \tag{15}$$

Finally, since

$$\mu^*_{r,0} = \mathscr{N}(0, \sigma I) \qquad \text{with} \quad \sigma \geqslant c_1 m^{-(r+1/2)}$$

for some constant c_1 (see [14]), this and (15) imply that

$$\lambda^{(p)}_{n,\delta}(C^r, L_\infty, \omega_r) \geqslant \lambda^{(p)}_{n,\delta}(\mathbb{R}^m, l^m_\infty, \mu^*_{r,0}) \geqslant c_1 m^{-(r+1/2)} \lambda^{(p)}_{n,\delta}(\mathbb{R}^m, l^m_\infty, \gamma).$$

Hence, Lemma 1 with $m = 2n$ completes the proof of the lower bound (14).

We now prove the upper bound,

$$\lambda^{(p)}_{n,\delta}(C^r, L_\infty, \omega_r) \leqslant c n^{-(r+1/2)} \sqrt{\ln(n/\delta)} \tag{16}$$

for a positive constant c. For this end, we use the following general result; see [5]. Let $\{n_k\}_k$ be a sequence of nonnegative integers and $\{\delta_k\}_k$ be a sequence of reals from $[0, 1]$. If

$$n_k \leqslant 2^k, \qquad \sum_{k=0}^\infty n_k \leqslant n, \qquad \sum_{k=0}^\infty \delta_k \leqslant \delta, \qquad \forall k \geqslant 0, \tag{17}$$

then

$$\lambda^{(p)}_{n,\delta}(C^r, L_\infty, \omega_r) \leqslant \sum_{k=0}^\infty 2^{-(r+1/2)k} \lambda^{(p)}_{n_k,\delta_k}(\mathbb{R}^{2^k}, l^{2^k}_\infty, \gamma). \tag{18}$$

Without loss of generality, we can assume that $n=2^{k'}$. Consider

$$n_k=\begin{cases}2^k, & k<k'-1,\\ \lfloor n2^{k'-k}\rfloor, & k\geqslant k'-1,\end{cases} \qquad \delta_k=\begin{cases}0, & k<k'-1,\\ \delta 2^{k'-k}, & k\geqslant k'-1.\end{cases}$$

Obviously, $\{n_k\}_k$ and $\{\delta_k\}_k$ satisfy (17), and

$$\sum_{k=0}^{\infty} 2^{-(r+1/2)k}\lambda^{(p)}_{n_k,\,\delta_k}(\mathbb{R}^{2^k}, l_\infty^{2^k}, \gamma)$$
$$\leqslant c_1 \sum_{k=k'-1}^{\infty} 2^{-(r+1/2)k}\sqrt{\ln((2^k-n_k)/(\delta 2^{k'-k}))}$$
$$\leqslant c_2 2^{-(r+1/2)k'}\sqrt{\ln(2k'/\delta)}$$

for some positive constants c_1 and c_2. This proves (16) and, hence, completes the proof of Theorem 1. ▮

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